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ABSTRACT

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The problem under consideration deals with the optimization of a low-thrust spaceflight with an energy-storing device. A general variational problem is formulated, and analysis of optimal control is made and some solutions for the movement in a forceless field and the turn of a satellite orbit are cited.

Author

Under consideration is a variational problem dealing with the maximum /1* load carried in a low-thrust flight with an energy-storing device. The use of such a device in combination with a limited-power propulsion system was proposed in reference 1, where a preliminary estimate of the possible gain was given. The formulation of the general variational problem, an analysis of the optimal controls and some possible solutions are cited below.

1. The main circuit of the low-thrust propulsion system with an energy-storing device contains power source N, energy storage E and a propelling agent (fig. 1). The $0 \leq N \leq N_0$ power produced by the source may be channelled to

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* Numbers given in margin indicate pagination in original foreign text.

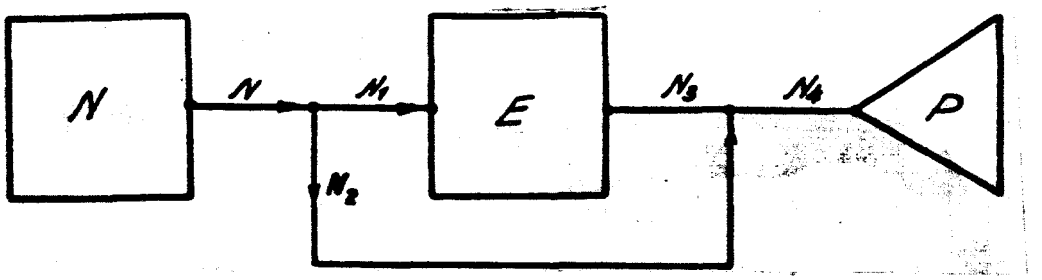

Figure 1

storage $N_1 \geq 0$ and directly to propelling agent $N_2 \geq 0$, so that $N_1 + N_2 = N$. The energy storage may deliver the power to engine $N_3 = N_1 - \dot{E}$($N_3 \geq 0$, $0 \leq E \leq E_0$; the point indicates time differentiation t). Thus the power $N_4 = N_2 + N_3$, which is used for producing thrust P by accelerating the working medium, flows to the propelling agent

$$P = qV, \quad N_4 = \tfrac{1}{2} qV^2 \quad (0 \leq P \leq P_0) \tag{1.1}$$

Here q represents the mass consumption of the working medium, and V /3 the outflow rate. The acceleration process of the working medium is assumed to be ideal, The thrust of the engine is limited from the top.

By excluding $N_2 = N - N_1$ and replacing $N_3 - N_1 = N_e$, it is possible to write the following equation of the power balance for the system under consideration

$$\dot{E} = -N_e, \quad N + N_e = \tfrac{1}{2} qV^2 \quad (0 \leq E \leq E_0, \; N_e \geq -N, \; 0 \leq N \leq N_0) \tag{1.2}$$

The negative values of power N_e correspond to the charging conditions and are limited by power N produced by the source. The positive values of N_e correspond to the discharging conditions. The weight of the low thrust propulsion system with the energy-storing device

$$G_x = G_y + G_e + G_\gamma \tag{1.3}$$

is made up of the following components[1] .

The weight of the power source (assumed to be proportional to the maximum power of source N_0) is

$$G_y = \alpha N_0 \tag{1.4}$$

The weight of the energy-storing device (assumed to be proportional to the maximum amount of stored energy E_0) is

$$G_e = \beta E_0 \tag{1.5}$$

The weight of the propelling device (assumed to be proportional to maximum thrust P_0) is

$$G_\gamma = \gamma P_0 \tag{1.6}$$

2. The problem of a maximum payload for a low-thrust propulsion /4 system with an energy-storing device is formulated as follows: initial weight G_0, specific weights α, β, γ, the dynamic maneuver and the time of maneuver T are defined; the object is to determine the optimal working conditions of the power source, the energy-storing device and propelling agent, as well as the optimal weight parameters (1.4)-(1.6) required for maximum payload G_π.

The complete system of differential equations, boundary conditions, restrictions on control functions and the phase coordinates of this variational

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[1] In reference 1 the weight G_x was assumed to be dependent only on power N_0.

problem, according to Myer's formulation, may be recorded as follows (similar to reference 2)

$$\dot{G}_\sigma = -\frac{k_1 P^2}{N(1+N_e)}, \quad G_\sigma(0) + G_x = 1, \quad G_\sigma(T) = \max$$

$$\dot{E} = -k_2 N N_e, \quad E(0) = 1, \quad E(T) = 0$$

$$\dot{\vec{r}} = \vec{v}, \quad \vec{r}(0) = \vec{r}_0, \quad \vec{r}(T) = \vec{r}_1$$

$$\dot{\vec{v}} = k_3 \frac{P\vec{\tau}}{G_\sigma + G_x} + \vec{R}, \quad \vec{v}(0) = \vec{v}_0, \quad \vec{v}(T) = \vec{v}_1 \qquad (2.1)$$

$$0 \le P(t) \le 1, \quad 0 \le N(t) \le 1, \quad N_e(t) \ge -1, \quad |\vec{\tau}(t)| = 1, \quad 0 \le E(t) \le 1$$

$$\left(k_1 = \frac{q P_0^2}{2 N_0 G_0} = \frac{\alpha q G_x^2}{2\gamma^2 G_v}, \quad k_2 = \frac{N_0}{E_0} = \frac{\beta G_v}{\alpha G_e}, \quad k_3 = \frac{q P_0}{G_0} = \frac{q G_x}{\gamma}\right).$$

Here the first equation is one of the consumption of working medium /5 [$G_\sigma(t) = G_\pi + G_\mu(t)$ is the total weight of payload G_π and fuel reserve G_μ], the second is an equation of the changing stored energy, and the third and fourth are vector equations of the mass-flow center of the apparatus in the inertial coordinate system [$\vec{R} = \vec{R}(\vec{r}, t)$ is the vector of acceleration from the gravitational forces].

The initial condition for weights G_σ and G_x corresponds to the given starting weight G_0 (all weights are referred to G_0). At the end of the movement, weight $G_\sigma(T)$ coincides by definition of G_π, and should be maximal. The boundary conditions for energy E (referred to E_0), radius vector $\vec{r}$ and velocity vector $\vec{v}$ may vary with the type of maneuver.

The control functions P, N, N_e and $\vec{i}$ are dimensionless: thrust P and the power of source N refer to their maximum values P_0 and N_0, power N_e applies to the current power of the source, and $\vec{i}$ is unit vector of the thrust direction.

The control parameters G_ν, G_e and G_γ--representing the weight of the power source, energy-storing device and propelling agent (and referred to G_0)--are included in the right side of equations (2.1) and the initial condition of the given starting weight (2.1). The selection of the optimal values of these parameters can be reduced to a standard procedure by following reference 3 and joining to system (2.1) the following formal equations

$$\dot{G}_\nu = 0, \quad \dot{G}_e = 0, \quad \dot{G}_\gamma = 0 \tag{2.2}$$

and boundary conditions

$$G_\sigma(0) + G_\nu(0) + G_e(0) + G_\gamma(0) = 1, \quad G_\nu(T) = \text{opt}, \quad G_e(T) = \text{opt}, \quad G_\gamma(T) = \text{opt} \tag{2.3}$$

(the first condition is substituted for $G_\sigma(o) + G_x = 1$),

The control parameters G_ν, G_e and G_γ thereupon become phase coordinates whose initial values are associated with transversality, and the finite values are free.

The optimal controls $\vec{i}(t)$, P(t), $N_e(t)$ and N(t) are defined by the Pontryagin method (ref. 4) on the basis of the absolute minimum of the Hamiltonian function

$$H = -P_\sigma \frac{k_1 P^2}{N(1+N_e)} - P_e k_2 N N_e + (\vec{P}_v \cdot \vec{i}) \frac{k_3 P}{G_\sigma + G_x} + (\vec{P}_r \cdot \vec{R}) + (\vec{P}_v \cdot \vec{v})$$

$$\left(\dot{P}_\sigma = -\frac{\partial H}{\partial G_\sigma}, \quad \dot{P}_e = -\frac{\partial H}{\partial E}, \quad \dot{\vec{P}}_v = -\frac{\partial H}{\partial \vec{v}}, \quad \dot{\vec{P}}_r = -\frac{\partial H}{\partial \vec{r}} \right) \tag{2.4}$$

A distinction should be made in this connection between the two types of trajectory sections: those within the $0 \leq E \leq 1$ region and those on the $E = 0$ or $E = 1$ boundaries. Omitting the intermediate stages of estimating the minimum function H, we will cite the final expressions for the optimal equations.

The laws governing the time changes of the thrust direction and power of source N are the same for both the internal and boundary sections of the trajectory,

$$\vec{i} = -\vec{P}_v / P_v, \quad N = 1 \qquad (P_v = |\vec{P}_v|) \tag{2.5}$$

that is the thrust vector is antiparallel to the $\vec{P}_v$ vector, and the power of the source is at a maximum (as well as in the low thrust propulsion system without and energy-storing device (ref. 5)).

Power N_e (controlling the stored energy) and thrust P are governed by /7 the following laws:

within the region

$$0 \leq E \leq 1$$

$$N_e = \sqrt{\frac{P_0 k_1}{P_e k_2}} - 1 \qquad P = 1 \quad \text{with} \quad P_v^2 > \varphi_1 \tag{2.6}$$

$$N_e = -1, \quad P = 0 \quad \text{with} \quad P_v^2 < \varphi_1$$

$$\left(\varphi_1 = 4 \frac{k_1 k_2}{k_3^2} P_0 P_e (G_x + G_0)^2\right)$$

on the E = 0 and E = 1 boundaries

$$N_e=0,\ P=1 \quad \text{with} \quad p_v^2 > \varphi_2 \left(\varphi_2 = 4\frac{k_1^2}{k_3^2} p_\sigma^2 (G_\sigma + G_x)^2\right)$$
$$N_e=0,\ P=\frac{p_v k_3}{2k_1 p_\sigma (G_\sigma+G_x)} \quad \text{with} \quad p_v^2 < \varphi_2 \qquad (2.7)$$

(the P_σ and P_e impulses are everywhere negative).

Thus the stored energy within the $0 \le E \le 1$ region is either used up at an optimally changing rate (discharging conditions, whereby the thrust is maximal--see first correlation of 2.6), or is stored with maximum velocity (charging, with zero thrust) see second correlation of (2.6)). On the E = 0 and E = I boundaries the consumption of the stored energy is naturally nil, and the thrust changes the same way as in the absence of an energy-storing device (ref. 2): it is regulated maximally (first correlation of (2.7)) or optimally (second correlation of (2.7)).

The behavior of the P_σ, P_e, $P_v = |\vec{P}_v|$ impulses and the G_σ phase /8 coordinate must be known before the alternation of optimal laws (2.6) and (2.7) can be established. The P_σ and P_v impulses and the phase coordinates are everywhere continuous, and the P_e, when drifting away from E = 0 and E = 1 boundaries because of the jump (ref. 4), has a discontinuity of the first kind (the form of function H and the differential equations for impulses (2.4) remain unchanged).

Taking (2.5) into consideration, we find from equations (2.4) and (2.1)

$$\dot{p}_\sigma = -\frac{p_v k_3 P}{(G_\sigma+G_x)^2} \le 0, \quad \dot{p}_e = 0, \quad \dot{G}_\sigma = -\frac{k_1 P^2}{1+N_e} \le 0 \qquad (2.8)$$

It is possible to show that the P_σ and P_e impulses are everywhere negative, and the φ_1 and φ_2, included in the (2.6) and (2.7) of the changing conditions, are positive. (2.8) facilitates the conclusion that these functions are nondecreasing

$$\dot{\varphi}_1 > 0,\ \dot{\varphi}_2 > 0 \text{ with } p_v^2 \geqslant \varphi_1;\ \dot{\varphi}_1 = 0,\ \dot{\varphi}_2 = 0 \text{ with } p_v^2 \leq \varphi_1$$

$$\dot{\varphi}_1 > 0,\ \dot{\varphi}_2 > 0 \text{ with } p_v^2 \geqslant \varphi_2;\ \dot{\varphi}_1 = 0,\ \dot{\varphi}_2 > 0 \text{ with } p_v^2 \leq \varphi_2$$

The behavior of impulse $\vec{P}_v$ is in the main determined by the type of maneuver. Characteristic of the examples under consideration hereafter is the nonmonotone behavior of $P_v^2(t)$ with a minimum within the interval [0, T].

If energy E fails to reach the E = 0 and E = 1 boundaries in the entire course of the movement, the E(t) and P(t) functions will appear as indicated in figure 2. In general, there may be some sections of movement along the E = 0 and E = 1 boundaries, respectively (fig. 3), during the change from dis- /9 charging conditions, and vice versa. The moment of drift from the boundary is not defined by the maximum principle, and should be found by varying the length of the boundary section.

Concluding the review of this item, we should like to emphasize the necessity of discussing the propulsion system with an energy-storing device but without a power source. Such a system may be found advantageous at low values of β, the specific weight of the energy-storing device (1.5). In this case the equations of the variational problem (2.1) are written as follows ($G_v = 0$)

$$\dot{G}_\sigma = -\frac{k_1 P^2}{N_e},\quad c_\sigma(0) + G_x = 1,\quad G_\sigma(T) = \max$$
$$\dot{E} = -k_2 N_e,\quad E(0) = 1,\quad E(T) = 0 \qquad (2.9)$$

$$\dot{\vec{r}} = \vec{v}, \quad \vec{r}(0) = \vec{r}_0, \quad \vec{r}(T) = \vec{r}_1,$$
$$\dot{\vec{v}} = k_3 \frac{P\vec{\tau}}{G_0 + G_x} + \vec{R}, \quad \vec{v}(0) = \vec{v}_0, \quad \vec{v}(T) = \vec{v}_1, \qquad (2.9)$$
$$0 \leq P \leq 1, \quad N_e(t) \geq 0, \quad |\vec{\tau}(t)| = 1$$
$$\left(k_1 = \frac{qP_0^2}{2E_0G_0} = \frac{q\beta G_x^2}{2\gamma^2 G_e}, \quad k_2 = \frac{1}{T}, \quad k_3 = \frac{qP_0}{G_0} = \frac{qG_x}{\gamma}\right).$$

Here N_e, unlike (2.1), applies to E_0/T, and there is no restriction on phase coordinate E (this restriction becomes automatic when the boundary conditions for E, with $N_e \geq 0$, are satisfied).

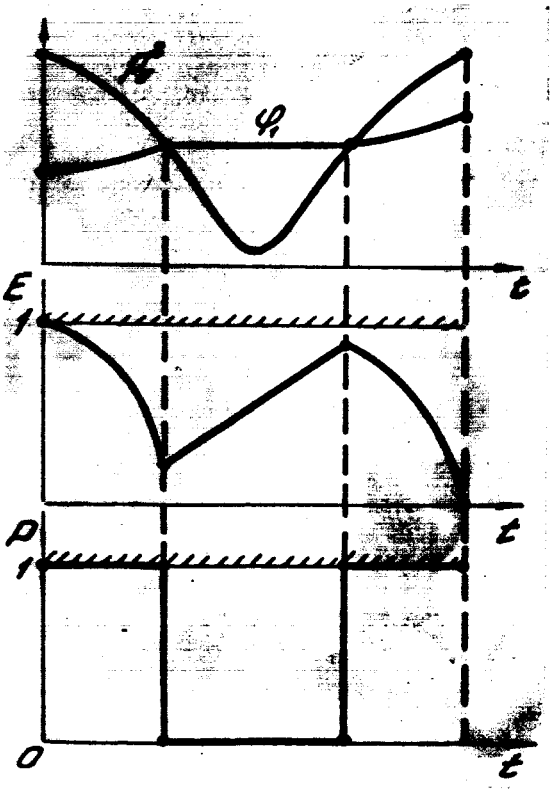

Figure 2

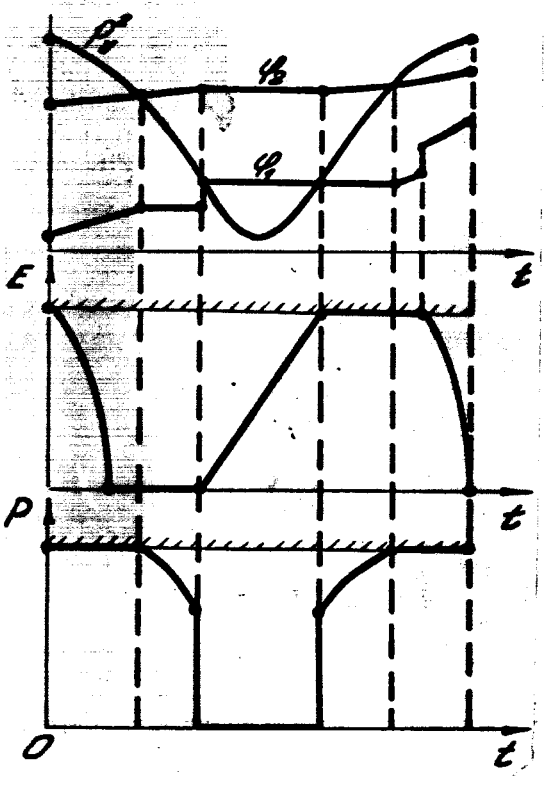

Figure 3

The following optimal laws apply to the control functions (compare (2.5) and (2.6)) /10

$$\vec{\tau} = -\vec{p}_v / p_v$$

$$N_e = \sqrt{\frac{p_\sigma k_1}{p_e}}, \quad P = 1 \text{ при } p_v^2 > \varphi_1 \quad \left(\varphi_1 = \frac{4k_3}{k_2^2}(G_\sigma + G_x)^2 p_\sigma p_e\right)$$

$$N_e = 0, \quad P = 0 \text{ при } p_v^2 < \varphi_1 \qquad (2.10)$$

3. A perfectly controlled propulsion system is characterized by the lack of a restriction on the thrust from above ($P \geq 0$) and zero weight of the propelling agent ($G_\gamma = 0$). A consideration of this case makes it possible to establish the upper limit to the system's possibilities, and divide the general variational problem into its weight and trajectory components.

The absence of a restriction on the thrust from above facilitates a change from thrust P to jet acceleration in equations (2.1)

$$a = k_3 P \quad (G_\sigma + G_x)^{-1} \qquad (3.1)$$

The first of the equations in (2.1) can then be integrated in quadratures and resolved in relation to the functional of the problem (just as in the case of the low thrust propulsion system without an energy-storing device (ref. 5))

$$G_\Pi = \left(1 - \frac{1}{G_\gamma}\frac{\alpha}{2q}\int_0^T \frac{a^2 dt}{N(1+N_e)}\right)^{-1} - G_\nu\left(1 + \frac{G_e}{G_\nu}\right) \qquad (3.2)$$

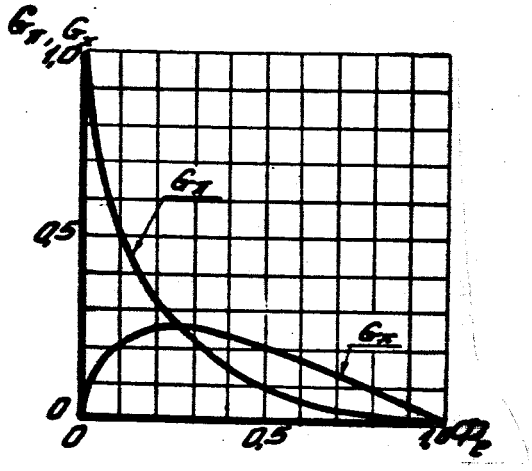

Figure 4

The relative weight of power source G_ν is included in the other equations /11 of variational problem (2.1) only in the form of the G_ν/G_e factor. It is therefore possible to find the optimal G_ν value ensuring a maximum G_π by recording the integral and G_e/G_ν relation in (3.2) before the solution of the full problem

$$G_\nu = \frac{1}{1+\xi}\left(\sqrt{\Phi_e} - \Phi_e\right)$$

$$\left(\xi = \frac{G_e}{G_\nu},\ \Phi_e = (1+\xi)\frac{\alpha}{2g}\int_0^T \frac{a^2\,dt}{N(1+N\ldots}\right). \qquad (3.3)$$

in this connection

$$G_\pi = (1-\sqrt{\Phi_e})^2,\ G_\varkappa = G_\nu + G_e = \sqrt{\Phi_e} - \Phi_e \qquad (3.4)$$

The latter correlations (fig. 4) for maximum payload G_π and optimal full weight of propulsion system $G_\varkappa$ coincide with the known solutions for a propulsion

system without an energy-storing device (ref. 5). But the following is substituted for φ_e in that correlation

$$\varphi = \frac{\alpha}{2g}\int_0^T a^2 dt$$

We should point out that in the problem involving an energy-storing device the optimal full weight of the propulsion system does not exceed the initial G_0 weight by 25 percent.

The payload in (3.4) decreases monotonically with increasing φ_e, and the second part of the problem is therefore reduced to determining the minimum of the functional (parameter α is prescribed)

$$\mathcal{J}_e = (1+\xi)\int_0^T \frac{a^2 dt}{N(1+N_e)} \qquad (3.5)$$

The effective utilization of the energy-storing device will be characterized by the relation of the (3.5) functional to /12

$$\mathcal{J} = \int_0^T a^2 dt \qquad (3.6)$$

If $0 < \mathcal{J}_e/\mathcal{J} < 1$, the use of the energy-storing device becomes advantageous.

The equations of the variational problem of the $\mathcal{J}_e$ minimum are recorded as follows

$$\dot{J}_e = \frac{(1+\xi)a^2}{N(1+N_e)}, \quad J_e(0)=0, \quad J_e(T)=\min$$

$$\dot{E} = -\frac{\beta}{\xi} NN_e, \quad E(0)=1, \quad E(T)=0$$

$$\dot{\vec{r}} = \vec{v}, \quad \vec{r}(0)=\vec{r}_0, \quad \vec{r}(T)=\vec{r}_1 \qquad (3.7)$$

$$\dot{\vec{v}} = a\vec{i} + \vec{R}, \quad \vec{v}(0)=\vec{v}_0, \quad \vec{v}(T)=\vec{v}_1$$

$$a(t)>0, \quad 0<N(t)\leq 1, \quad N_e(t)\geq -1, \quad |\vec{i}(t)|=1, \quad 0\leq E(t)\leq 1$$

These are the optimal controls (compare (2.5)-(2.7))

$$\vec{i} = \vec{p}_v / p_v, \quad N=1, \quad a = p_v(1+N_e)/2(1+\xi)$$

$$N_e = \infty \text{ with } p_v^2 > 4\frac{\beta(1+\xi)}{\alpha\xi} p_e \text{ и } 0<E<1 \ (p_e>0) \qquad (3.8)$$

$$N_e = 0 \text{ with } p_v^2 < 4\frac{\beta(1+\xi)}{\alpha\xi} p_e \text{ и } 0<E<1; \text{ with } E\equiv 0; \text{ with } E=1$$

that is, the discharging conditions are of the impulse type. /13

In the case of an extreme case of a propulsion system without a source of power, the (3.4) correlations remain in force, assuming that

$$G_v = 0, \quad \Phi_e = \frac{\beta T}{2g}\int_0^T \frac{a^2 dt}{N_e}$$

(the N_e power applies to E_0/T).

The equations of variational problem (3.7) change in the same manner as at the end of the previous paragraph, and the following becomes the functional

$$J_e = \int_0^T \frac{a^2 dt}{N_e}$$

Two maneuvers for which the equation of the variational problem are integrated to the end are considered in connection with the solution of a general problem: I^0. The shift between points of rest in a forceless field (ref. 5), and 2^0. The turn of the circular orbit plane by a lateral thrust in a central gravitational field (refs. 6 and 7):

I^0. In the case of a shift between points of rest ($V_0 = V_1 = 0$) located at a distance of 1 ($\tau_0 = 0$, $\tau = 1$), during period T, P_v is a linear time function (ref. 5), that is, $P_v^2 = (b - ct)^2$. If there is no energy-storing device, functional (3.6) in this problem equals (ref. 5)

$$J_l = 12 l^2 / T^3 \qquad (3.9)$$

As in point 2, the optimal programs E(t), $N_e(t)$ and a(t) are constructed on the basis of correlations (3.8) with some $P_v^2(t)$ function. These programs are shown in figure 5 (where $t_{1,2} = 1/2\ [T \pm \xi\ d/\beta t]$). /14

Functional (3.5) is found after equations (3.7) have been integrated by section, and the boundary conditions have been satisfied

$$J_e = J_l \frac{1+\xi}{1+6x-x^3} \quad \left(x = \frac{\alpha}{\beta T} \xi \right)$$

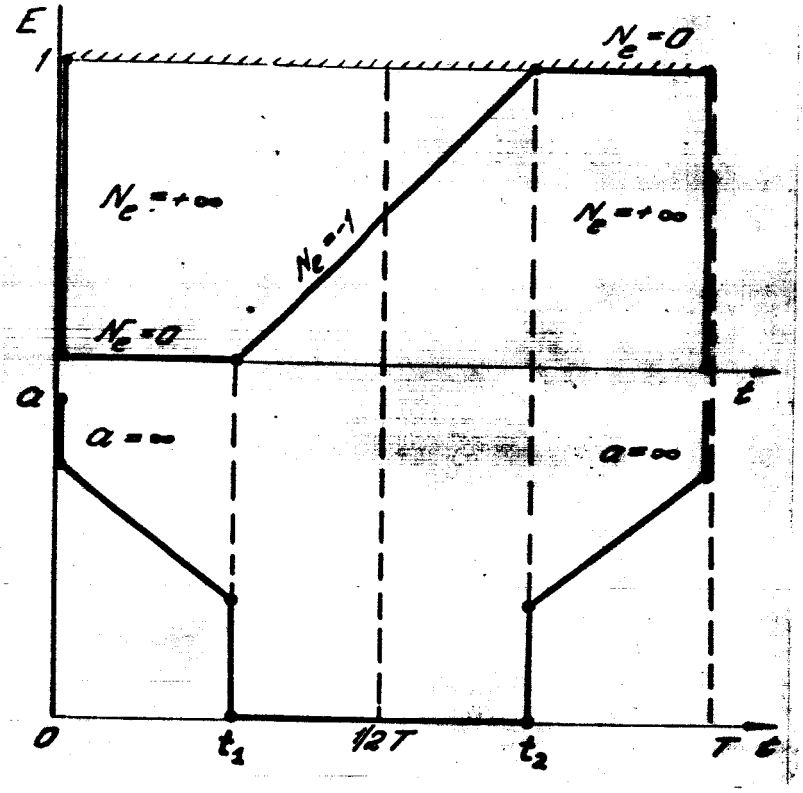

Figure 5

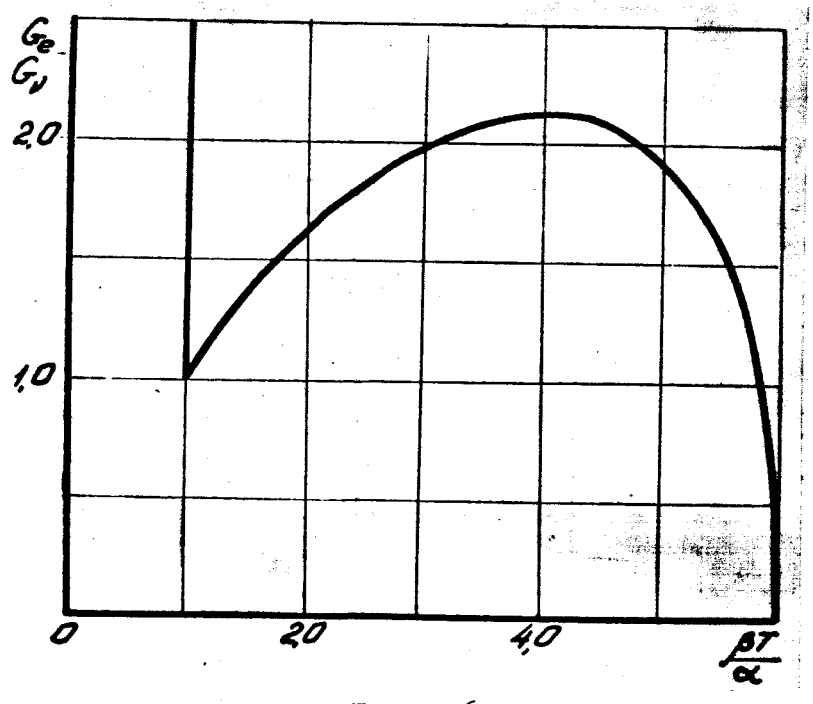

Figure 6

The minimum of that expression, based on $\xi = G_e/G_v$, is achieved when

$$\frac{\beta T}{\alpha} = \frac{3(2-x^2)}{1+2x^3} \tag{3.10}$$

then

$$\Phi_e/\Phi_l = (1+2x^3)^{-1} \tag{3.11}$$

Shown in figure 6 is dependence (3.10) of the optimal G_e/G_v relation of the weight of the energy-storing device to the weight of the source from dimensionless parameter $\beta T/\alpha$ (β represents the specific weight of the energy-storing device (1.5), and α the specific weight of source of power (1.4) and T is the time of motion, that is the maximum possible charging time). The decrease in functional Φ during the transition to a system with a storage device is shown in figure 7. The payload gain is determined by the use of that figure on the basis of the first (3.4) formula, or by figure 4.

A low-thrust propulsion system with an energy-storing device has an advantage over a system without a storing device within the range of $I \leq \beta T/\alpha \leq 6$. With $0 \leq \beta T/\alpha \leq 1$, the advantage lies with the system equipped with an energy-storing device but without a source of power ($G_v = 0$ in figure 6) for which

$$\Phi_e/\Phi_l = 1/3\ \beta T/\alpha$$

2^o. When radius r_o of the circular orbit plane is turned to angle Ψ /15 in an n number of revolutions, P_v by a cosine (ref. 7), that is $P_v^2 = \xi \cos t$. (time t applies to $T_* = \sqrt{r_o/g_o}$, the revolution period divided into 2π, and g_o is the gravitational acceleration on radius r_o). In the case of a perfect

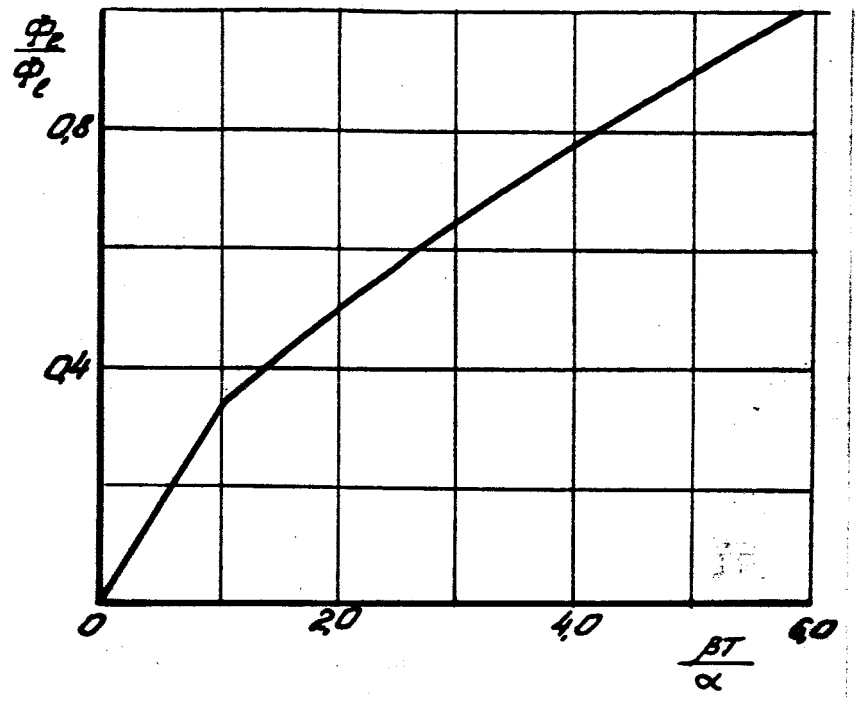

Figure 7

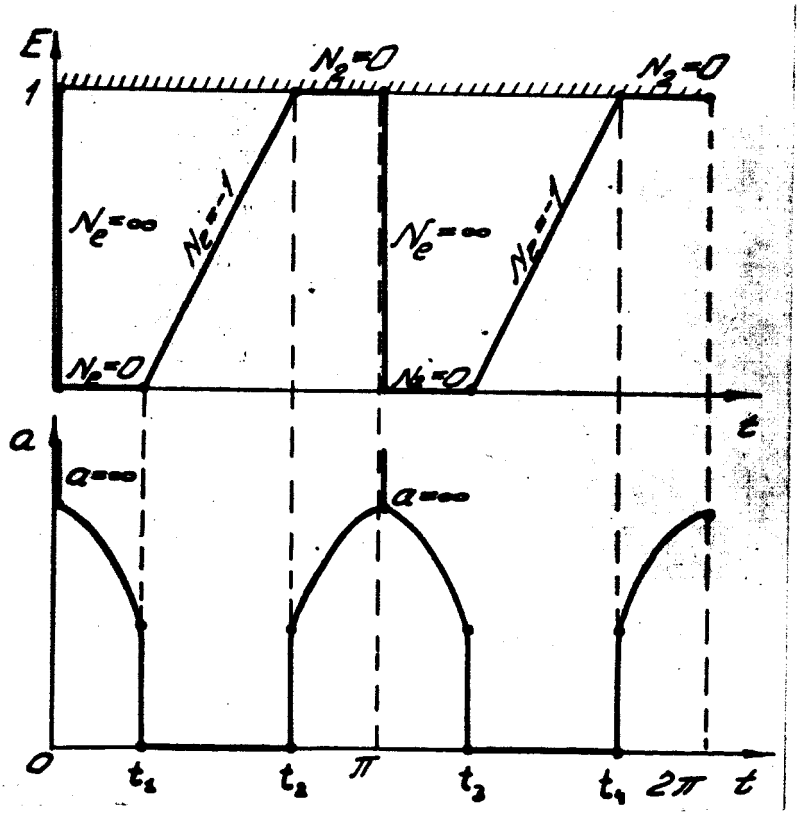

Figure 8

low-thrust propulsion system without an energy-storing device, the (3.6) functional equals (ref. 7)

$$J_\psi = \frac{r_0^2}{T^3}\frac{\psi^2}{n\pi}. \qquad (3.12)$$

In the problem involving an energy-storing device it is possible to show that the optimal laws governing the changes of $E(t)$, $N_e(t)$ and $a(t)$ are the periodic time functions recurring at every half turn. The nature of their change in one revolution is shown in figure 8 (where $t_{1,2} = 1/2\,(\pi \mp \xi\alpha/\beta T)$, $t_{3,4} = t_{1,2} + \pi$).

The formulas for the optimal ξ value and minimum Φ_e/Φ_ψ relation are similar to (3.10) and (3.11)

$$\frac{\beta T_*}{\alpha} = \frac{1+\cos x + 1/n}{\pi + \sin x - x\cos x}; \quad \frac{\Phi_e}{\Phi_\psi} = \frac{\pi}{\pi + \sin x - x\cos x} \qquad (3.13)$$

The dependence of G_e/G_ν and Φ_e/Φ_ψ on $\beta\pi T_*/\alpha$ (πT_* is the time of half a revolution, that is the maximum charging time) for various n values is shown in figures 9 and 10.

A low-thrust propulsion system with an energy-storing device is advantageous in the range of $1/2n \le \beta\pi T_*/\alpha \le 2 + 1/n$ and a propulsion system with a storing device without a source of power is more suitable within the range of

• In this connection

$$\Phi_e/\Phi_\psi = n(\beta\pi T_*/\alpha)$$

(see linear sections in figure 10).

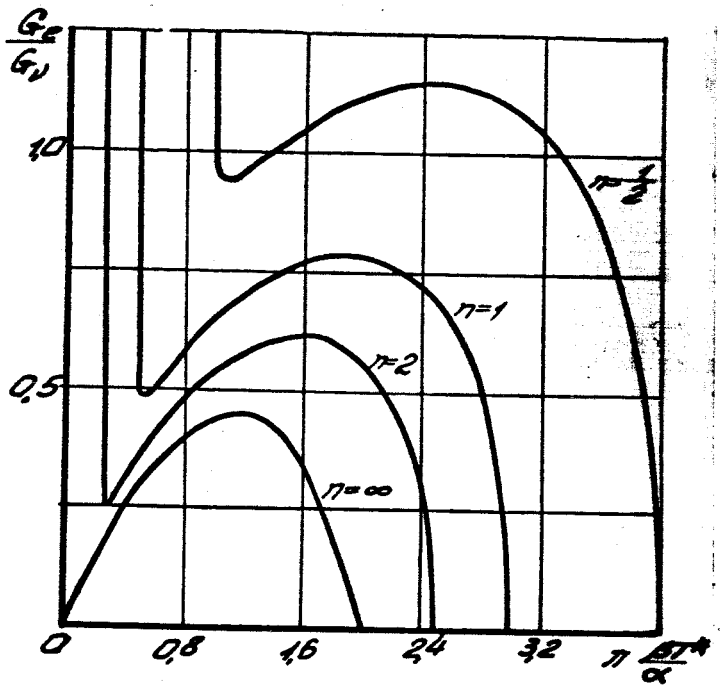

Figure 9

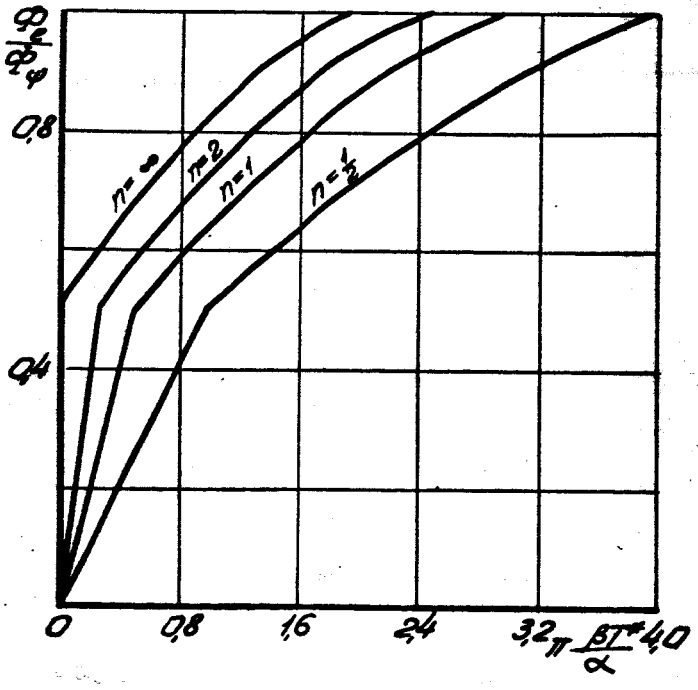

Figure 10

The gain in payload, as in the case of I°, is determined by the use /16 of figure 10, on the basis of the first (3.4) formula, or by figure 4.

4. An uncontrolled propulsion system is characterized by a thrust of a constant magnitude (and constant power of the jet stream). A consideration of this second extreme case reveals the lower limit of the system's possibilities.

Here the form of the (2.1) differential equations is somewhat different (in the first two equations)

$$\dot{G}_0 = -k_1 P/N_*, \quad \dot{E} = k_2[N(1-P)-(N_*-1)P] \tag{4.1}$$

Control function P(t) can have only two values: zero and 1 (and in this case the exponent of P in (2.1) may be omitted). The power N_* used by the propelling device is constant, representing the control parameter (applies to N). A discharge at a constant rate $\dot{E} = -K_2(N - 1)$ takes place on the powered-flight sections (P = 1), and a charge $\dot{E} = K_2N$ at a rate of $0 \leq N(t) \leq I$ occurs on the coasting sections (P = 0).

The equation of motion is recorded as before, and control function $\vec{i}(t)/ \equiv I$ (direction of the thrust vector) is preserved.

For purposes of simplification, the constancy of jet acceleration (3.1) should be substituted for that of the thrust. This is valid if the relative weight of working medium is considerably less than 1. Furthermore, the weight of the propelling device, unlike (1.6), is taken to be proportional to the employed power $G_\gamma = \gamma_* N_* N_0$.

By such an approximation it is possible, as in the previous paragraph, /17 to find the optimal weight of the source of power before the final solution of

the problem. Payload G_{π} and the weight of the propulsion system $G_x = G_{\gamma} + G_e + G_{\gamma}$ will be expressed by formulas (3.4) where

$$\Phi_e = (1 + \xi + \frac{\gamma_*}{\alpha} N_*) \frac{\alpha}{2g} \int_0^T \frac{a^2 dt}{N_*} \tag{4.2}$$

In a system without a storing-device, functional Φ (taking the engine weight $G_{\gamma} = \gamma_* N_*$ into account) will equal

$$\Phi^* = (1 + \frac{\gamma_*}{\alpha}) \frac{\alpha}{2g} \int_0^T a^2 dt \tag{4.3}$$

In (4.2) and (4.3) $a^2(t) = a_0^2 P(t)$, $P(t) = 1$ or 0.

The variational problem is written down as in (3.7).

$$\begin{aligned}
&\dot{J}_e = \left(\frac{1+\xi}{N_*} - \frac{\gamma_*}{\alpha}\right) a_0^2 P, \quad J_e(0) = 0, \quad J_e(T) = \min \\
&\dot{E} = \frac{\beta}{\alpha \xi} [N(1-P) - (N_* - 1)P], \quad E(0), \quad E(T) = 0 \\
&\dot{\vec{r}} = \vec{v}, \quad \vec{r}(0) = \vec{r}_0, \quad \vec{r}(T) = \vec{r}_1 \\
&\dot{\vec{v}} = a_0 P \vec{i} + \vec{R}, \quad \vec{v}(0) = \vec{v}_0, \quad \vec{v}(T) = \vec{v}_1 \\
&P(t) = 1 \text{ or } 0, \quad 0 \le N(t) \le 1, \quad |\vec{i}(t)| = 1, \quad 0 \le E(t) \le 1
\end{aligned} \tag{4.4}$$

In addition to control functions P(t), N(t) and $\vec{i}(t)$, the problem includes the following three constant control parameters: α_0, N_*, ξ. The example

cited in point 2 can be used for the selection of the optimal values of these parameters.

We will cite the solution for maneuvers 1° and 2° without writing down the composition of the optimal controls.

1°. In the problem involving the shift between the points of rest, /18 the relation between functionals (4.2) and (4.3) is

$$\frac{\Phi_e}{\Phi_e^{\circ}} = \frac{4[\varepsilon(1-4t+2t^2]}{27t^2(1-4t)(1-t)^2(1+\varepsilon)} \qquad (\varepsilon = \gamma_*/\alpha) \tag{4.5}$$

where optimal parameters ξ and t (instead of $a_0 = t^{-1}(1-t)^{-1}$, $N^* = (1-t)t^{-1}$) are determined from the following correlations

$$\xi = \frac{\beta T}{\alpha}(1-2t), \quad \frac{\beta T}{\alpha} = \frac{\varepsilon(1-t)(1-3t)-3t^2}{t^2(1-4t)} \tag{4.6}$$

With $(\beta T/\alpha) < (\beta T/\alpha)^*$ the use of a system without a source of power appears to be more profitable.

$$\frac{\Phi_e}{\Phi_e^{\circ}} = \frac{4[2(\beta T/\alpha)+\xi]}{27t(1-t^2)(1+\varepsilon)} \qquad (2t^2-(\beta T/\alpha)(1-3t)=0) \tag{4.7}$$

The dependence of weight relations G_e/G_ν and functionals Φ_e/φ_e° on $\beta T/\alpha$ for various $\varepsilon = \gamma_*/\alpha$ values (the relation of the specific engine weight and source of power) is shown in figures 11 and 12.

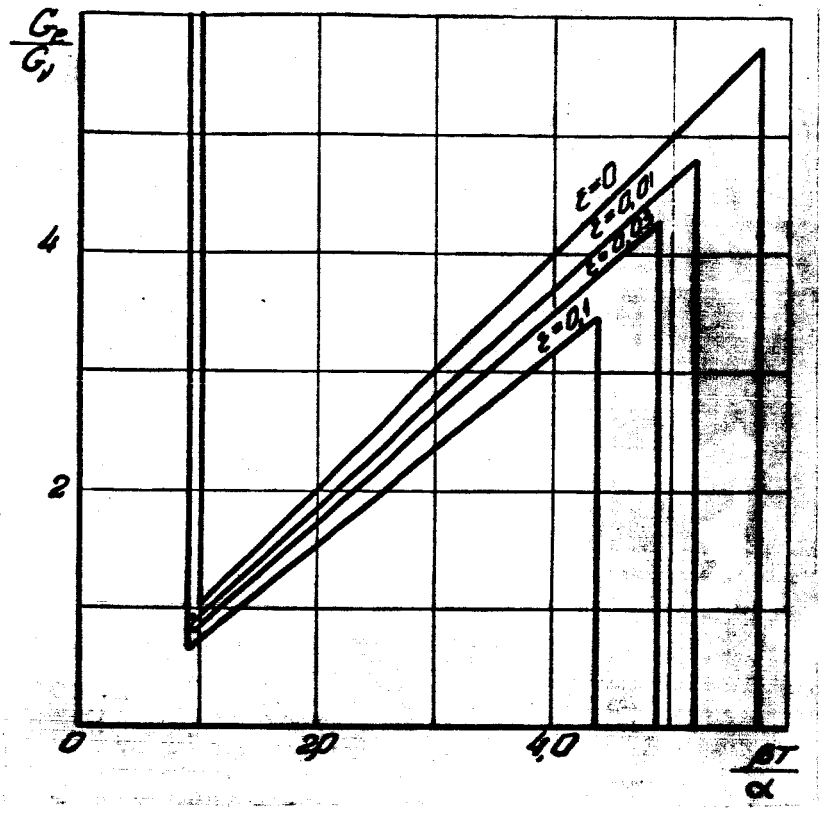

Figure 11

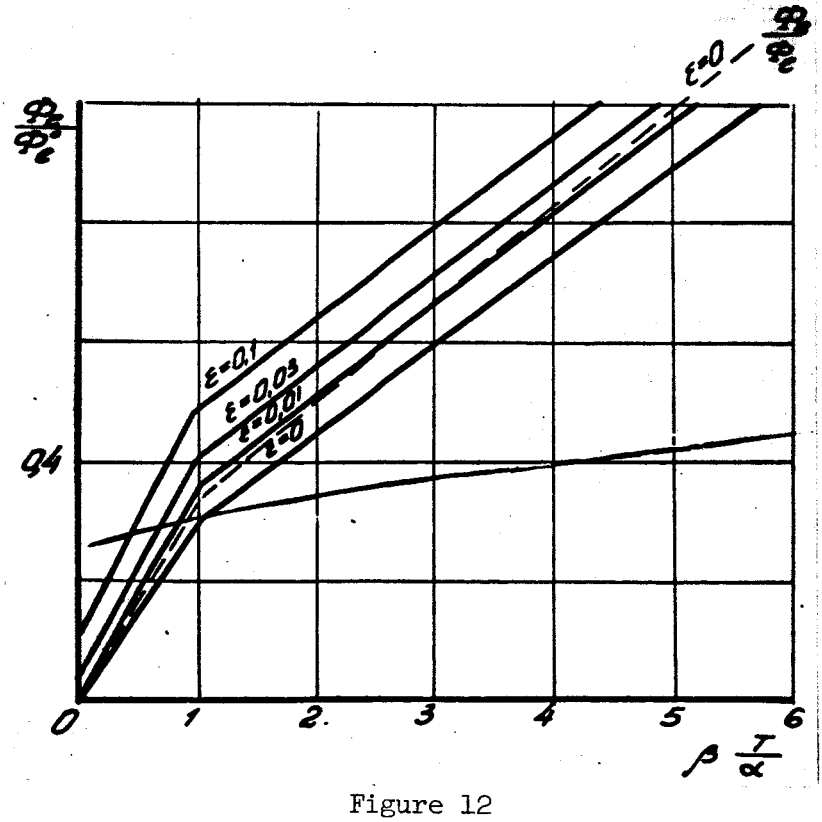

Figure 12

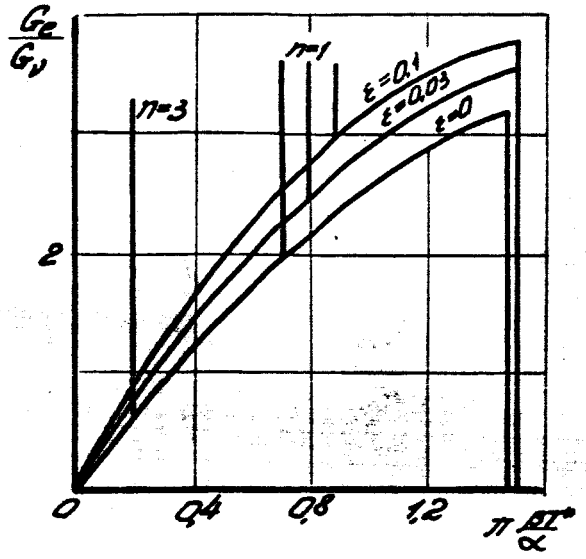

Figure 13

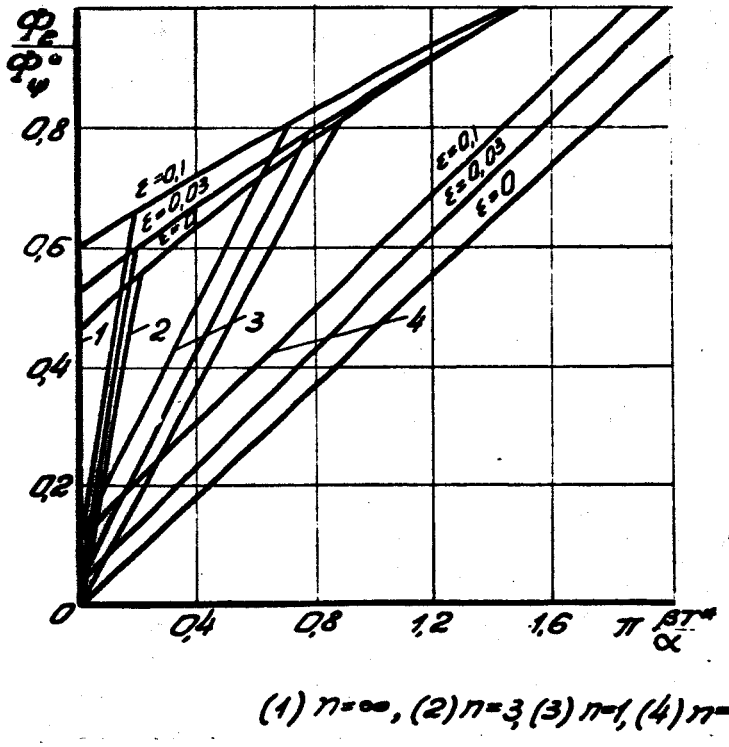

Figure 14

In the problem under consideration the value of functional τ is (ref. 5) (compare (3.9))

$$\mathcal{J}_\ell^\circ = 13{,}5\,\ell^2/T^3 \qquad (4.8)$$

(constant jet acceleration and switching off).

A dotted curve in figure 12 indicates correlation ϕ_e/φ_e with $\gamma_* /\alpha = 0$ [19] (instead of Φ_e/Φ_e°) for purposes of comparison. A comparison of this dotted curve and figure 7 reveals that, as in the case of limited thrust motion without a storing device (ref. 5), the change from a perfectly controlled propulsion system to an a = const. system with a switch-off calls for a slight increase in functional ϕ. As parameter $\beta T/\alpha$ decreases, the difference disappears as both systems change to impulse working conditions.

2°. In the problem involving the turn of the orbital plane, the relation of the functional equals

$$\frac{\phi_e}{\varphi_\psi^\circ} = \frac{1{,}45\left[t^3 \operatorname{tg} t + \varepsilon\pi t \operatorname{tg} t\,(t-\pi/4)\right]}{\pi(1+\varepsilon)\sin^2 t\left[t(\pi-2t)-(\pi-3t)\operatorname{tg} t\right]}$$

$$\left(\xi = \frac{\beta T}{\alpha}(\pi-2t),\quad \frac{\beta T}{\alpha} = \frac{(t+1/4\,\varepsilon\pi)\operatorname{tg} t - 1/2\,\varepsilon\pi t - t^2}{t^2(\pi-2t)-(\pi-3t)\operatorname{tg} t t}\right.$$

$$\left. a_0 = \frac{\psi}{4n\sin t},\quad N_* = \frac{\pi}{2t}\right)$$

The system without a source of power has an advantage in the range of

$$0 < (\beta T/\alpha) \leq (\beta T/\alpha)^*$$

$$\frac{\Phi_\varepsilon}{\Phi_\psi^\circ} = \frac{0.3625\, t\, \varepsilon}{(1+\varepsilon)(\sin t - t\cos t)\sin t}$$

$$\left(\frac{\beta T}{L} = \varepsilon \frac{2t\cos t - \sin t}{8\pi t (\sin t - t\cos t}, \quad J_\psi^\circ = \frac{r_0^2 \psi^2}{T_*^3\, 2{,}9\pi} \right)$$

Figures 13 and 14, which are similar to figures 11 and 12, are constructed on the basis of these correlations. The payload gain the cases under consideration is determined, as above, by the use of figure 4 or the first formula of (3.4). /20

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